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On Certain Duffin and Schaeffer Type Inequalities*

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Duffin and Schaeffer type inequalities related to some ultraspherical polynomials are established. One of the results obtained reads as follows: Let f be a real algebraic polynomial of degree at most n, such that $|f(\pm 1)| \leqslant 1$ and $|f(x)| \leqslant \sqrt{1-x^2}$ at the zeros of $T_{n-1}(x)$. Then $\max_{-1 \leqslant x \leqslant 1} |f^{(k)}(x)| \leqslant T_n^{(k)}(1)$ for all $k \in \{1, ..., n\}$. Moreover, equality holds if and only if $f = \pm T_n$.

1. INTRODUCTION

Answering a question of the prominent Russian chemist D. Mendeleev, in 1890 A. A. Markov proved that if $f(x) = \sum_{i=0}^n a_i x^i$ is a real algebraic polynomial of degree at most n such that $|f(x)| \leqslant 1$ in $[-1, 1]$, then in the same interval

$$|f'(x)| \leqslant n^2.$$

Two years later, in 1892, A. Markov's younger brother V. A. Markov (being at that time a student at St. Petersburg University) extended this result proving the following

THEOREM A. *If $f(x) = \sum_{i=0}^n a_i x^i$ is a real algebraic polynomial of degree not exceeding n and $|f(x)| \leqslant 1$ in $[-1, 1]$, then*

$$\max_{x \in [-1, 1]} |f^{(k)}(x)| \leqslant \frac{n^2(n^2-1^2)\cdots(n^2-(k-1)^2)}{1.3\cdots(2k-1)} = T_n^{(k)}(1) \qquad (1.1)$$

for $k = 1, ..., n$. Equality holds only for $f(x) = \pm T_n(x) = \pm\cos(n \arccos x)$.

Inequalities of the brothers Markov type have been a challenge for many mathematicians. In 1941 Duffin and Schaeffer [4] strengthened Theorem A

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proving that inequality (1.1) remains true if the requirement $|f(x)| \leqslant 1$ in $[-1, 1]$ is replaced by

$$|f(\eta_j^n)| \leqslant 1, \qquad j = 0, 1, ..., n, \tag{1.2}$$

where $\eta_j^n = \cos(j\pi/n)$ are the points of local extrema of $T_n(x)$ in $[-1, 1]$. In addition, Duffin and Schaeffer showed that (1.1) fails to hold if the conditions (1.2) are replaced by $f|_E \leqslant 1$, where E is any closed set of points in $[-1, 1]$ which does not contain all the points $\{\eta_j^n\}$. In fact Duffin and Schaeffer proved a more general result including inequality for polynomials in a strip in the complex plane, but this result does not fall in the frame of this paper. We only mention that their proof involves complex arguments, in particular the Rouche theorem.

Denote by π_n the class of all real algebraic polynomials of degree not exceeding n, and by $\mathscr{P}_n$ the subset of π_n containing only polynomials with n distinct real zeros, located in $(-1, 1)$. In our notation Q_n will mean a given algebraic polynomial of exact degree n (we call it *majorant*), and $\|f\| := \sup_{x \in [-1, 1]} |f(x)|$. We now formulate our definition for Duffin and Schaeffer type inequality (DS-inequality).

Duffin and Schaeffer Type Inequality. The polynomial Q_n and the mesh $\Delta = \{t_j\}_{j=0}^n$ $(-1 = t_0 < t_1 < \cdots < t_n = 1)$ are said to admit DS-inequality if for an arbitrary $f \in \pi_n$ the assumptions $|f(t_j)| \leqslant |Q_n(t_j)|$ $(j = 0, 1, ..., n)$ imply the inequalities $\|f^{(k)}\| \leqslant \|Q_n^{(k)}\|$ for $k = 1, 2, ..., n$ (in some cases we prove this only for $k \geqslant 2$ or for $k \geqslant 3$).

Note that the inequalities of DS-type do not hold unconditionally. The validity of such inequalities depends on the choice of the majorant Q_n and on the mesh Δ. Actually, to the best of our knowledge, only a few DS-inequalities of the above mentioned type are hitherto known.

In 1970 P. Turán raised the following question (see [10]):

Problem. If $f \in \pi_n$ satisfies the inequalities

$$|f(x)| \leqslant \sqrt{1 - x^2} \qquad \text{for} \quad -1 \leqslant x \leqslant 1, \tag{1.3}$$

then, how large can $\|f^{(k)}\|$ be?

This question was answered in [10] for the case $k = 1$, and in [9] (the general case). The extremal polynomial turned out to be

$$Q_n(x) = (x^2 - 1)\, U_{n-2}(x), \tag{1.4}$$

where $U_m(x) = \sin[(m+1) \arccos x]/\sqrt{1 - x^2}$ denotes the mth Chebyshev polynomial of second kind.

Subsequently, it was proved by Rahman and Schmeisser [12] that the polynomial (1.4) remains extremal with respect to $\|f^{(k)}\|$ in a larger class of polynomials than those defined by (1.3). Namely, they proved the following DS-type inequality.

THEOREM B. *If f is an algebraic polynomial of degree at most n, satisfying the inequality*

$$|f(x)| \leqslant \sqrt{1-x^2} \text{ at the zeros of } (x^2-1)\, T_{n-1}(x),$$

then

$$\|f^{(k)}\| \leqslant Q_n^{(k)}(1) \tag{1.5}$$

for all $k \in \{2, ..., n\}$ and

$$\|f'\| \leqslant (n-1)\left(\frac{2}{\pi}\log(n-1)+3\right) = \frac{2}{\pi}(1+o(1))\, n\log n$$

as $n \to \infty$. Further, in (1.5) *equality holds only if $f(x) = \gamma Q_n(x)$ where $|\gamma| = 1$.*

Note that Theorem B is true for complex-valued polynomials.

In a recent paper A. Shadrin [14] turned back to the original idea of V. Markov—Lagrange interpolation. He presented a simple non-complex proof of Theorem A under assumptions (1.2). The crucial part for his proof is

THEOREM C. *Let $q \in \mathscr{P}_n$, and let $t_j = t_j(q)$ $(j = 0, ..., n)$ be the points of all local extrema of q in $[-1, 1]$. Suppose that $f \in \pi_n$ and*

$$|f(t_j)| \leqslant |q(t_j)|, \qquad j = 0, ..., n.$$

Then, for every $x \in [-1, 1]$ and for $k = 1,...,n$,

$$|f^{(k)}(x)| \leqslant \max\left\{|q^{(k)}(x)|, \left|\frac{1}{k}(x^2-1)\, q^{(k+1)}(x) + xq^{(k)}(x)\right|\right\}.$$

Shadrin has conjectured that DS-inequality holds for every $Q_n \in \mathscr{P}_n$ provided the mesh $\varDelta$ is taken to contain the points of local extrema of Q_n in $[-1, 1]$, i.e., if $\varDelta = \{-1\} \cup \{t\colon Q_n'(t) = 0\} \cup \{1\}$. Unfortunately, as some simple examples show, this conjecture is not true in general. Nevertheless, using Theorem C, Bojanov and Nikolov [2] proved that DS-type inequality holds for such a choice of $\varDelta$ with majorant $Q_n = P_n^{(\lambda)}$ —the ultraspherical polynomial (the polynomial, orthogonal in $[-1, 1]$ with respect to the weight $(1-x^2)^{\lambda-1/2}$).

THEOREM D. *Let* $t_j := t_j(P_n^{(\lambda)})$ $(j=0, ..., n)$ *be the extremal points of* $P_n^{(\lambda)}$ *in* $[-1, 1]$. *Let* $f \in \pi_n$ *satisfy*

$$|f(t_j)| \leqslant |P_n^{(\lambda)}(t_j)|, \qquad j=0, ..., n.$$

Then the inequality

$$\|f^{(k)}\| \leqslant \left\| \frac{d^k}{dx^k} P_n^{(\lambda)} \right\|$$

holds for all $k \in \{1, ..., n\}$, *if* $\lambda \geqslant 0$, *and for all* $k \in \{2, ..., n\}$, *if* $-1/2 \leqslant \lambda < 0$.

A very interesting result (though not exactly of DS-type) is established in [3]. There, inequalities for the norms of the derivatives of polynomials are found on the basis of a comparison of their corresponding local extrema.

We prove in this paper some new DS-type inequalities with majorants Q_n as in Theorems B and D. Section 2 contains some preliminary results. In Section 3 we extend the pointwise inequality given by Theorem C (Theorems 3.1–3.3). Precisely, starting from a fixed mesh Δ we obtain a family of polynomials which may serve as majorants in DS-type inequalities related to Δ. In Section 4 we apply this extension to obtain DS-type inequalities for $Q_n = P_n^{(\lambda)}$ with $\Delta = \{t_j\}_{j=0}^{n}$, $t_0 = -1$, $t_n = 1$, and $\{t_j\}_1^{n-1}$ being the zeros of $P_{n-1}^{(\lambda)}$ (Theorems 4.1–4.2). In Section 5 we establish DS-type inequalities for a similar choice of Δ but for majorants that vanish at the points -1 and 1 (Theorems 5.1–5.3). Section 6 contains some comments and remarks.

2. AUXILIARY RESULTS

The following two lemmas belong to V.A. Markov and reveal the very interesting fact, that if two polynomials have only real simple zeros, which interlace, then the interlacing property remains valid also for their derivatives.

LEMMA 2.1. *Let* $b_1 > b_2 > \cdots > b_{s+1}$; $c_1 > c_2 > \cdots > c_s$, *and let* $b_1 \geqslant c_1 \geqslant b_2 \cdots \geqslant c_s \geqslant b_{s+1}$. *Let* $p(t) = \prod_{i=1}^{s}(t-c_i)$ *and* $q(t) = \prod_{i=1}^{s+1}(t-b_i)$.

Then for $1 \leqslant k \leqslant s-1$ *the zeros of* $p^{(k)}(t)$: $\gamma_1 > \gamma_2 > \cdots > \gamma_{s-k}$ *and the zeros of* $q^{(k)}(t)$: $\beta_1 > \beta_2 > \cdots > \beta_{s+1-k}$ *interlace, i.e., satisfy the inequalities*

$$\beta_1 > \gamma_1 > \beta_2 > \cdots > \beta_{s-k} > \gamma_{s-k} > \beta_{s+1-k}.$$

LEMMA 2.2. *Let* $b_1 > b_2 > \cdots > b_s$; $c_1 > c_2 > \cdots > c_s$, *and let* $b_1 \geqslant c_1 \geqslant b_2 \cdots \geqslant c_s$ *with* $b_j \neq c_j$ *for at least one j. Let* $p(t) = \prod_{i=1}^{s} (t - c_i)$ *and* $q(t) = \prod_{i=1}^{s} (t - b_i)$.

Then for $1 \leqslant k \leqslant s-1$ *the zeros of* $p^{(k)}(t)$: $\gamma_1 > \gamma_2 > \cdots > \gamma_{s-k}$ *and the zeros of* $q^{(k)}(t)$: $\beta_1 > \beta_2 > \cdots > \beta_{s-k}$ *satisfy the inequalities*

$$\beta_1 > \gamma_1 > \beta_2 > \cdots > \beta_{s-k} > \gamma_{s-k}.$$

As is pointed out by Bojanov in [1, p. 39], the assertion of Lemma 2.2 could be regarded also as monotone dependence of the zeros of the derivative with respect to the zeros of the polynomial. For the proof of Lemmas 2.1–2.2 the reader may refer to [14] or Rivlin's book [11, Lemma 2.7.1].

The next lemma summarizes some observations of V.A. Markov concerning the pointwise estimates for derivatives of a polynomial. Its proof is based on Lemmas 2.1–2.2 and the Lagrange interpolation formula (see, e.g., [14, Lemma 2]).

LEMMA 2.3. *Let* $\omega_* \in \mathscr{P}_{n-1}$, *and let* $\{t_j\}_{j=0}^{n}$ *be the ordered zeros of* $\omega(x) = (x^2 - 1)\, \omega_*(x)$. *Let* $Q_n \in \pi_n$ *satisfy* $Q_n(t_{j-1})\, Q_n(t_j) < 0$ *for* $j = 1, ..., n$. *If f is a polynomial of degree at most n satisfying the inequalities*

$$|f(t_j)| \leqslant |Q_n(t_j)| \qquad \text{for} \quad j = 0, ..., n, \tag{2.1}$$

then for every $k \in \{1, ..., n\}$ *there exists a set* $I_{n,k} = I_{n,k}(\omega)$, *such that*

$$|f^{(k)}(x)| \leqslant |Q_n^{(k)}(x)| \qquad \text{for all} \quad x \in I_{n,k}. \tag{2.2}$$

The set $I_{n,k}$ *is given by*

$$I_{n,k} = [-1, \alpha_1^k] \cup [\beta_1^k, \alpha_2^k] \cup \cdots \cup [\beta_{n-k-1}^k, \alpha_{n-k}^k] \cup [\beta_{n-k}^k, 1], \tag{2.3}$$

where $\{\alpha_j^k\}_1^{n-k}$ *and* $\{\beta_j^k\}_1^{n-k}$ *are the ordered zeros of* $\omega_0^{(k)}$ *and* $\omega_n^{(k)}$, *respectively, and* $\omega_j(x) = \omega(x)/(x - t_{n-j})$.

Moreover, if equality occurs in (2.2) *for some* $x \in I_{n,k}$, $x \neq \alpha_i^k, \beta_i^k$ $(i = 1, ..., n-k)$, *then* $f = \gamma Q_n$, *where* $|\gamma| = 1$.

REMARK 1. The conditions $Q_n(t_{j-1})\, Q_n(t_j) < 0$ for $j = 1, ..., n$ can be replaced by the weaker requirement that the zeros $\{\theta_j\}_1^n$ of Q_n interlace with the zeros of ω, i.e., to satisfy the inequalities $t_0 \leqslant \theta_1 \leqslant t_1 \leqslant \cdots \leqslant \theta_n \leqslant t_n$. Thus, Q_n can be allowed to have a zero at ± 1, and then ω and f must vanish at this point, too. Moreover, Lemma 2.3 remains true if the first and the last intervals in (2.3) are replaced by $(-\infty, \alpha_1^k]$ and $[\beta_{n-k}^k, \infty)$. Denote by $J_{n,k} := J_{n,k}(\omega)$ the complementery set of $I_{n,k}(\omega)$,

$$J_{n,k} = [-1, 1] \backslash I_{n,k} = \bigcup_{j=1}^{n-k} (\alpha_j^k, \beta_j^k).$$

By analogy with the notation in [6], we will call $I_{n,k}$ and $J_{n,k}$ Chebyshev and Zolotarev intervals, respectively. As was mentioned by Shadrin, for $k=n$ $J_{n,k}=\emptyset$, and for $k=n-1$ $|f^{(k)}(x)|$ attains its maximum at $x=-1$ or $x=1$, i.e., at a point from $I_{n,n-1}$. Therefore for $k=n-1, n$ the assumptions (2.1) imply $\|f^{(k)}\| \leqslant \|Q_n^{(k)}\|$ [14, Corollary 4]. For this reason we may assume in what follows $n \geqslant 3$.

Next, we list some properties of the ultraspherical polynomials $P_n^{(\lambda)}$ which will be needed for the proofs of Theorems 4.1 and 5.1.

Properties:

(i) $y = P_n^{(\lambda)}$ satisfies the differential equation

$$(1-x^2)\,y'' - (2\lambda+1)\,xy' + n(n+2\lambda)\,y = 0;$$

(ii) for $\lambda > 0$, $\|P_n^{(\lambda)}\| = |P_n^{(\lambda)}(\pm 1)|$;

(iii) $(d/dx)\,P_n^{(\lambda)}(x) = 2\lambda P_{n-1}^{(\lambda+1)}(x)$ $(\lambda \neq 0)$;

(iv) $(d/dx)\,P_{n+1}^{(\lambda)}(x) = x(d/dx)\,P_n^{(\lambda)}(x) + (n+2\lambda)\,P_n^{(\lambda)}(x)$ $(\lambda \neq 0)$;

(v) for $\lambda > 0$ the ultraspherical polynomials obey the representation

$$P_n^{(\lambda)}(x) = \sum_{m=0}^{n} a_{n,m}(\lambda)\, T_m(x)$$

with positive coefficients $a_{n,m}(\lambda)$.

Usually, the parameter λ is required to satisfy $\lambda > -1/2$; however in our theorems we allow also $\lambda = -1/2$. With respect to this case, we recall that $P_n^{(-1/2)}(x)$ is equal, apart from a constant factor, to $(1-x^2)(d/dx)\,P_{n-1}^{(1/2)}(x)$.

The proof of these properties can be found in the book of Szegö [16] (concerning property (v), the reader can find a more general statement in [11, p. 158, Remark 1]).

We conclude this section with a lemma, based on property (v).

LEMMA 2.4. *Let $q = P_n^{(\lambda)}$, $\lambda \geqslant 0$. Then for $k = 1, 2, ..., n$ and for every $s \geqslant k$*

$$\left\| \frac{x^2-1}{s}\, q^{(k+1)}(x) + xq^{(k)}(x) \right\| = |q^{(k)}(\pm 1)|. \tag{2.4}$$

For $\lambda \in [-1/2, 0)$ equality (2.4) holds for $k = 2, ..., n$.

Proof. We apply the approach proposed in [2]. Instead of (2.4) we shall show that for all $x \in [-1, 1]$ and for $\lambda \geqslant 0$

$$|(x^2-1)\, q^{(k+1)}(x) + sxq^{(k)}(x)| \leqslant sq^{(k)}(1). \tag{2.5}$$

In the case $s=k$ and $q=T_n$ (i.e., for $\lambda=0$) (2.5) has already been proved by Shadrin [14, Lemma 3]. Then, for $\lambda>0$, we make use of properties (v) and (ii) to obtain

$$\begin{aligned}
&|(x^2-1)\, q^{(k+1)}(x) + kxq^{(k)}(x)| \\
&\quad = \left|(x^2-1) \sum_{m=0}^{n} a_{n,m}(\lambda)\, T_m^{(k+1)}(x) + kx \sum_{m=0}^{n} a_{n,m}(\lambda)\, T_m^{(k)}(x)\right| \\
&\quad \leqslant \sum_{m=0}^{n} a_{n,m}(\lambda)\, |(x^2-1)\, T_m^{(k+1)}(x) + kxT_m^{(k)}(x)| \\
&\quad \leqslant \sum_{m=0}^{n} a_{n,m}(\lambda)\, kT_m^{(k)}(1) = kq^{(k)}(1),
\end{aligned}$$

proving in such a way (2.5) for $s=k$. For $s>k$ we have

$$\begin{aligned}
&|(x^2-1)\, q^{(k+1)}(x) + sxq^{(k)}(x)| \\
&\quad \leqslant |(x^2-1)\, q^{(k+1)}(x) + kxq^{(k)}(x)| + |(s-k)\, xq^{(k)}(x)| \\
&\quad \leqslant kq^{(k)}(1) + (s-k)\, q^{(k)}(1) = sq^{(k)}(1).
\end{aligned}$$

In the last step we have taken into account that, according to (iii) $q^{(k)}$ is an ultraspherical polynomial, too, and therefore in view of (ii) for $x \in [-1, 1]$ $|xq^{(k)}(x)| \leqslant q^{(k)}(1)$.

Finally, for the case $\lambda \in [-1/2, 0)$ one can apply the above arguments to $q'(x) = 2\lambda P_{n-1}^{(\lambda+1)}(x)$. The proof of lemma is completed. ▮

Remark 2. In [2] the same reasoning is applied for the proof of Theorem D, the case $\lambda \geqslant 0$, while the proof of the case $\lambda \in [-1/2, 0)$ relies on different arguments. Lemma 2.4 furnishes a short proof of Theorem D for both cases. Neither in Lemma 2.4 nor in Theorem D is the real situation known when $k=1$ and $\lambda \in [-1/2, 0)$.

3. POINTWISE INEQUALITIES

The proof of our DS-inequalities is based on some pointwise inequalities, established in this section. The main result is given in the next theorem.

THEOREM 3.1. *Let $\omega_* \in \mathcal{P}_{n-1}$, and let $Q_n \in \pi_n$ have n distinct zeros, which interlace with the zeros of $(x^2-1)\,\omega_*(x)$. Let for a $k \in \{1, ..., n\}$ $Q_n^{(k)}$ have a representation*

$$Q_n^{(k)}(x) = c_1 \omega_*^{(k-1)}(x) + c_2 x \omega_*^{(k)}(x) \tag{3.1}$$

with some constants c_1 and c_2, such that

$$\begin{aligned} c_1(c_1 - kc_2) > 0, &\qquad \text{if} \quad 1 \leqslant k \leqslant n-2, \\ (c_1+c_2)(c_1 - kc_2) > 0, &\qquad \text{if} \quad k = n-1, \\ c_1, c_2 \text{ arbitrary}, &\qquad \text{if} \quad k = n. \end{aligned} \tag{3.2}$$

If $f \in \pi_n$ satisfies the inequality

$$|f(x)| \leqslant |Q_n(x)| \text{ at the zeros of } (x^2-1)\,\omega_*(x), \tag{3.3}$$

then for every $x \in [-1, 1]$

$$|f^{(k)}(x)| \leqslant max\{|Q_n^{(k)}(x)|, |Z_{n,k}(x)|\},$$

where

$$Z_{n,k}(x) = (c_1 - kc_2)\left[\frac{x^2-1}{k}\,\omega_*^{(k)}(x) + x\omega_*^{(k-1)}(x)\right] - c_2\omega_*^{(k)}(x). \tag{3.4}$$

Proof. We consider first the main case $1 \leqslant k \leqslant n-2$. Without loss of generality we may assume that ω_* has a positive leading coefficient. Denote $\omega_0(x) := (x+1)\,\omega_*(x)$, $\omega_n(x) := (x-1)\,\omega_*(x)$, and let $\{\alpha_j^k\}_{j=1}^{n-k}$ and $\{\beta_j^k\}_{j=1}^{n-k}$ be the ordered zeros of $\omega_0^{(k)}$ and $\omega_n^{(k)}$, respectively. Lemma 2.2 shows that each interval (α_j^k, β_j^k) $(j=1, ..., n-k)$ contains exactly one zero of $Q_n^{(k)}$. Analogously, Lemma 2.1 asserts that the ordered zeros $\{\gamma_j^k\}_{j=1}^{n-k-1}$ of $\omega_*^{(k)}$ satisfy $\gamma_j^k \in (\beta_j^k, \alpha_{j+1}^k)$ $(j=1, ..., n-k-1)$. Therefore we obtain

$$\text{sign } Q_n^{(k)}(\gamma_{n-k-1}^k) = \text{sign}\{c_1\omega_*^{(k-1)}(\gamma_{n-k-1}^k)\} = -\text{sign } c_1,$$

and since $Q_n^{(k)}$ has exactly one zero located to the right from γ_{n-k-1}^k, we conclude that c_1 and the leading coefficient of Q_n have the same sign. Then (3.2) shows that the same sign has the main coefficient in $Z_{n,k}$. It is easily seen that

$$Z_{n,k}(x) - Q_n^{(k)}(x) = \frac{c_1 - kc_2}{k}\,(x - x_0)\,\omega_0^{(k)}(x), \tag{3.5}$$

$$Z_{n,k}(x) + Q_n^{(k)}(x) = \frac{c_1 - kc_2}{k}\,(x + x_0)\,\omega_n^{(k)}(x), \tag{3.6}$$

where $x_0 = c_1/(c_1 - kc_2)$. We therefore have for $j = 1, ..., n-k$

$$Z_{n,k}(x) = \begin{cases} Q_n^{(k)}(x) & \text{for} \quad x = \alpha_j^k, \\ -Q_n^{(k)}(x) & \text{for} \quad x = \beta_j^k. \end{cases} \tag{3.7}$$

In particular, the last relation yields

$$\operatorname{sign} Z_{n,k}(\beta_j^k) = -\operatorname{sign} Z_{n,k}(\alpha_{j+1}^k) \qquad \text{for} \quad j = 1, ..., n-k-1. \tag{3.8}$$

Moreover, since

$$Z_{n,k}(\beta_{n-k}^k) - Q_n^{(k)}(\beta_{n-k}^k) = -2Q_n^{(k)}(\beta_{n-k}^k),$$

we have

$$\operatorname{sign}[Z_{n,k} - Q_n^{(k)}](\beta_{n-k}^k) = -\operatorname{sign} c_1. \tag{3.9}$$

On the other hand,

$$\operatorname{sign}[Z_{n,k} - Q_n^{(k)}] = \operatorname{sign} c_1 \qquad \text{for sufficiently large } x;$$

therefore x_0 is the last zero of $Z_{n,k} - Q_n^{(k)}$, i.e., $x_0 > \beta_{n-k}^k$. Analogously, $-x_0 < \alpha_1^k$.

Now let $f \in \pi_n$ be an arbitrary polynomial satisfying (3.3), then according to Lemma 2.3 the kth derivatives of f and Q_n satisfy the inequalities

$$|f^{(k)}(x)| \leqslant |Q_n^{(k)}(x)| \qquad \text{for all} \quad x \in I_{n,k}. \tag{3.10}$$

The theorem will be proved if we show that

$$|f^{(k)}(x)| \leqslant |Z_{n,k}(x)| \qquad \text{for all} \quad x \in J_{n,k}. \tag{3.11}$$

From (3.7) and (3.10), for $j = 1, ..., n-k$ we get

$$|f^{(k)}(\alpha_j^k)| \leqslant |Z_{n,k}(\alpha_j^k)|, \tag{3.12}$$

$$|f^{(k)}(\beta_j^k)| \leqslant |Z_{n,k}(\beta_j^k)|. \tag{3.13}$$

This coupled with (3.8) yields

$$(Z_{n,k} \pm f^{(k)})(\beta_j^k) \cdot (Z_{n,k} \pm f^{(k)})(\alpha_{j+1}^k) \leqslant 0;$$

therefore $Z_{n,k} \pm f^{(k)}$ has at least one zero in $[\beta_j^k, \alpha_{j+1}^k]$ for $j = 1, ..., n-k-1$. The same observation applies to the intervals $[-x_0, \alpha_1^k)$ and $(\beta_{n-k}^k, x_0]$. We show this only for $(\beta_{n-k}^k, x_0]$; the second case is analogous. Since x_0 is the last zero of $Z_{n,k}-Q_n^{(k)}$, we have $|Z_{n,k}(x)| \geqslant |Q_n^{(k)}(x)|$ for $x \geqslant x_0$. On the other hand, $|Q_n^{(k)}(x)| \geqslant |f^{(k)}(x)|$ for $x \geqslant \beta_{n-k}^k$; therefore

$$\operatorname{sign}\{Z_{n,k}(x) \pm f^{(k)}(x)\} = \operatorname{sign} c_1 \qquad \text{for} \quad x \geqslant x_0,$$

while

$$\operatorname{sign}\{(Z_{n,k} \pm f^{(k)})(\beta_{n-k}^k)\} = -\operatorname{sign} Q_n^{(k)}(\beta_{n-k}^k) = -\operatorname{sign} c_1,$$

whence the desired result holds. Thus, we proved that each of the polynomials $Z_{n,k} \pm f^{(k)}$ has at least $n-k+1$ distinct zeros, located outside the Zolotarev intervals $J_{n,k}$. Since $Z_{n,k} \pm f^{(k)}$ are of exact degree $n-k+1$, they do not vanish on $J_{n,k}$. Then inequality (3.11) holds by virtue of (3.12)–(3.13). The theorem is proved in the case $1 \leqslant k \leqslant n-2$.

When $k=n-1$, we make use of the fact that the sign of the leading coefficient of Q_n is equal to $\operatorname{sign}\{c_1+c_2\}$; then (3.2) shows that the same sign has the leading coefficient of $Z_{n,n-1}$. Repeating the above reasoning, we conclude that if $f \in \pi_n$ satisfies (3.3), then each of the polynomials $Z_{n,n-1} \pm f^{(n-1)}$ contains a zero in $[-x_0, \alpha_1^{n-1})$ and a zero in $(\beta_1^{n-1}, x_0]$; hence $Z_{n,n-1} \pm f^{(n-1)}$ do not vanish on $J_{n,n-1}=(\alpha_1^{n-1}, \beta_1^{n-1})$, and then (3.12)–(3.13) imply $|f^{(n-1)}(x)| \leqslant |Z_{n,n-1}(x)|$ for $x \in J_{n,n-1}$.

The case $k=n$ is trivial, since, as was mentioned in Remark 1, in this case $|f^{(n)}(x)| \leqslant |Q^{(n)}(x)|$ for all $x \in (-\infty, \infty)$. ∎

Our next theorem asserts a condition under which pointwise inequalities hold for all $k \in \{1, ..., n\}$.

THEOREM 3.2. *Let $q \in \mathscr{P}_n$ and let $\{t_j\}_{j=1}^{n-1}$ be the zeros of q', $t_0 := -1$, $t_n := 1$. Let $Q_n(x) = mxq'(x) + q(x)$, where m is a real parameter such that*

$$m \geqslant \max\left\{\frac{q(-1)}{q'(-1)},\ -\frac{q(1)}{q'(1)}\right\}. \tag{3.14}$$

If $f \in \pi_n$ satisfies the inequalities

$$|f(t_j)| \leqslant |Q_n(t_j)| \qquad \text{for} \quad j=0, ..., n,$$

then for all $k \in \{1, ..., n\}$ and for every $x \in [-1, 1]$

$$|f^{(k)}(x)| \leqslant \max\{|Q_n^{(k)}(x)|, |Z_{n,k}(x)|\},$$

where

$$Z_{n,k}(x) = \left(\frac{x^2-1}{k} - m\right) q^{(k+1)}(x) + xq^{(k)}(x). \tag{3.15}$$

Proof. We only outline the differences from the proof of Theorem 3.1. Again, we may regard that the leading coefficient of q is positive. Then we show that the polynomial Q_n has exactly n real zeros, which interlace with the zeros of $\omega(x)=(x^2-1)\,q'(x)$. Indeed, under the assumptions of the theorem, sign $q(t_j)=(-1)^{n-j}$, $j=0, ..., n-1$; therefore

$$\operatorname{sign} Q_n(t_j)=(-1)^{n-j} \qquad \text{for} \quad j=0, ..., n-1, \tag{3.16}$$

and each of the intervals (t_{j-1}, t_j) $(j=2, ..., n-1)$ contains a zero of Q_n. Moreover, requirement (3.14) together with (3.16) implies the existence of two additional zeros of Q_n located in $[-1, t_1)$ and $(t_{n-1}, 1]$, respectively. Thus we established the desired interlacing property. In addition, it follows from (3.16) that Q_n has a positive leading coefficient. The same is true for $Z_{n,k}$, and it is easily seen that the polynomials $Z_{n,k} \pm Q_n^{(k)}$ obey the representations (3.5)–(3.6) with c_1-kc_2 replaced by 1 and $x_0=x_0(k)=1+km$. The proof then is completed in the same way as in Theorem 3.1. ▮

Remark 3. Requirement (3.14) is fulfilled, e.g., if $m \geqslant 0$. In the special case $m=0$ Theorem 3.2. reproduces Shadrin's Theorem C.

Theorem 3.2 treats the symmetric case only, but applying the same arguments as above one can extend it as follows

THEOREM 3.3. *Let $q \in \mathscr{P}_n$ and let $\{t_j\}_{j=1}^{n-1}$ be the zeros of q', $t_0 := -1$, $t_n := 1$. Let $Q_n(x)=(mx+s)\,q'(x)+q(x)$, where m and s are real parameters such that*

$$m-s \geqslant \frac{q(-1)}{q'(-1)}, \qquad m+s \geqslant -\frac{q(1)}{q'(1)}.$$

If $f \in \pi_n$ satisfies the inequalities

$$|f(t_j)| \leqslant |Q_n(t_j)| \qquad \textit{for} \quad j=0, ..., n,$$

then for all $k \in \{1, ..., n\}$ and for every $x \in [-1, 1]$

$$|f^{(k)}(x)| \leqslant \max\{|Q_n^{(k)}(x)|, |Z_{n,k}(x)|\},$$

where

$$Z_{n,k}(x)=\left(\frac{x^2-1}{k}-sx-m\right) q^{(k+1)}(x)+(x-ks)\,q^{(k)}(x).$$

4. INEQUALITIES OF DUFFIN AND SCHAEFFER TYPE

As an application of Theorem 3.1 we prove in this section a DS-type inequality where the majorant Q_n is the ultraspherical polynomial $P_n^{(\lambda)}$.

THEOREM 4.1. *Let $\{t_j\}_{j=1}^{n-1}$ be the zeros $P_{n-1}^{(\lambda)}$, $t_0 := -1$, $t_n := 1$. If $f \in \pi_n$ satisfies the inequalities*

$$|f(t_j)| \leqslant |P_n^{(\lambda)}(t_j)|, \qquad j=0, ..., n,$$

then

$$\|f^{(k)}\| \leqslant \left\| \frac{d^k}{dx^k} P_n^{(\lambda)} \right\| \tag{4.1}$$

for each $k \in \{1, ..., n\}$, if $\lambda \geqslant 1$, for $k \in \{2, ..., n\}$, if $\lambda \in [0, 1)$, and for $k \in \{3, ..., n\}$, if $\lambda \in (-1/2, 0)$.

For these values of k and λ, in (4.1) equality holds only if $f = \pm P_n^{(\lambda)}$.

Proof. Set $\omega_* := P_{n-1}^{(\lambda)}$, $Q_n := P_n^{(\lambda)}$. Then obviously the zeros of Q_n and $(x^2-1)\,\omega_*(x)$ interlace. Moreover, property (iv) and repeated differentiation yield

$$Q_n^{(k)}(x) = (n+2\lambda+k-2)\,\omega_*^{(k-1)}(x) + x\omega_*^{(k)}(x) \qquad \text{for} \quad k=1, ..., n. \tag{4.2}$$

For $n \geqslant 3$ the constants $c_1 = (n+2\lambda+k-2)$ and $c_2 = 1$ satisfy (3.2); therefore Theorem 3.1 is applicable and for $x \in [-1, 1]$ and $k \in \{1, ..., n\}$ there holds $|f^{(k)}(x)| \leqslant \max\{|Q_n^{(k)}(x)|, |Z_{n,k}(x)|\}$, where

$$Z_{n,k}(x) = (n+2\lambda-2)\left[\frac{x^2-1}{k}\,\omega_*^{(k)}(x) + x\omega_*^{(k-1)}(x)\right] - \omega_*^{(k)}(x). \tag{4.3}$$

Our goal is to show that $\|Q_n^{(k)}\| \geqslant \|Z_{n,k}\|$ for all cases of k and λ, postulated in the theorem. Based on property (ii), we find

$$\|Q_n^{(k)}\| = (n+2\lambda+k-2)\,|\omega_*^{(k-1)}(1)| + |\omega_*^{(k)}(1)| \tag{4.4}$$

for $k \geqslant 1$, if $\lambda > 0$, and for $k \geqslant 2$, if $\lambda \in [-1/2, 0)$.

Next, we apply Lemma 2.4 to obtain

$$\begin{aligned}\|Z_{n,k}\| &\leqslant (n+2\lambda-2)\left\|\frac{x^2-1}{k}\,\omega_*^{(k)}(x) + x\omega_*^{(k-1)}(x)\right\| + \|\omega_*^{(k)}\| \\ &= (n+2\lambda-2)\,|\omega_*^{(k-1)}(1)| + |\omega_*^{(k)}(1)| \end{aligned} \tag{4.5}$$

for $k \geqslant 1$, if $\lambda \geqslant 1$, for $k \geqslant 2$, if $\lambda \in (0, 1)$, and for $k \geqslant 3$, if $\lambda \in [-1/2, 0)$. Now comparison of the right-hand sides of (4.4) and (4.5) asserts the desired result. Theorem 4.1 is proved for $\lambda \neq 0$.

The proof of the case $\lambda = 0$ needs a slight modification due to the different normalization of the Chebyshev polynomials of first kind. We put $Q_n = (1/n)\, T_n$, $\omega_* = T_{n-1}$, and replace the identity (4.2) by

$$Q_n^{(k)}(x) = \frac{n+k-2}{n-1}\, T_{n-1}^{(k-1)}(x) + \frac{1}{n-1}\, x T_{n-1}^{(k)}(x) \tag{4.6}$$

to obtain from Theorem 3.1 $|f^{(k)}(x)| \leqslant \max\{|Q_n^{(k)}(x)|, |Z_{n,k}(x)|\}$ with

$$Z_{n,k}(x) = \frac{n-2}{n-1}\left[\frac{x^2-1}{k}\, T_{n-1}^{(k)}(x) + x T_{n-1}^{(k-1)}(x)\right] - \frac{1}{n-1}\, T_{n-1}^{(k)}(x) \tag{4.7}$$

Applying again Lemma 2.4, we obtain for $k \geqslant 2$

$$\begin{aligned} \|Z_{n,k}\| &\leqslant \frac{n-2}{n-1}\, T_{n-1}^{(k-1)}(1) + \frac{1}{n-1}\, T_{n-1}^{(k)}(1) \\ &\leqslant \frac{n+k-2}{n-1}\, T_{n-1}^{(k-1)}(1) + \frac{1}{n-1}\, T_{n-1}^{(k)}(1) = \|Q_n^{(k)}\|. \end{aligned}$$

Finally, the cases of equality are easily clarified on the basis of Lemma 2.3. The proof is completed. ∎

Remark 4. The proposed method of proof does not work in the cases $k = 1$, $\lambda \in [0, 1)$ and $k = 1, 2$, $\lambda \in [-1/2, 0)$; therefore the real situation in these cases is not known. Actually, it turns out that in the special case $\lambda = 0$ inequality (4.1) holds for $k = 1$, too. Since $Q_n = T_n$ seems to be the most important case, we formulate it in a separate theorem.

THEOREM 4.2. *Let f be a real algebraic polynomial of degree at most n, satisfying the inequalities*

$$|f(x)| \leqslant \sqrt{1-x^2} \qquad \text{at the zeros of } T_{n-1}(x), \tag{4.8}$$

and

$$|f(\pm 1)| \leqslant 1. \tag{4.9}$$

Then

$$\|f^{(k)}\| \leqslant T_n^{(k)}(1) \qquad \text{for all} \quad k \in \{1, ..., n\}. \tag{4.10}$$

Equality in (4.10) *is possible only if* $f = \pm T_n$.

Proof of Theorem 4.2. It remains to prove (4.10) for $k=1$, and we do this by showing that for

$$Z_{n,1}(x)=\frac{1}{n-1}\,[(n-2)[(x^2-1)\,T'_{n-1}(x)+xT_{n-1}(x)]-T'_{n-1}(x)]$$

the estimate

$$|Z_{n,1}(x)|\leqslant \|Q'_n\|=\frac{1}{n}\,\|T'_n\|=n \qquad (x\in[-1,1]) \tag{4.11}$$

is true. For reasons of symmetry we assume $x\in[0,1]$. Consider separately two cases.

I. The case $x\in[0,\xi]$, where ξ will be specified later. Due to the estimate

$$|T'_{n-1}(x)|\leqslant\frac{n-1}{\sqrt{1-x^2}},$$

we have

$$|Z_{n,1}(x)|<\frac{n-2}{n-1}\,|(x^2-1)\,T'_{n-1}(x)|+\frac{1}{n-1}\,|T'_{n-1}(x)|+1$$

$$\leqslant h(x):=(n-2)\sqrt{1-x^2}+\frac{1}{\sqrt{1-x^2}}+1.$$

The function $h(x)$ has in $(0,1)$ exactly one extremum which is a minimum; thus on $[0,\xi]$

$$h(x)\leqslant\max\{h(0),h(\xi)\}=\max\{n,h(\xi)\}.$$

We choose $\xi\in(0,1)$ such that $h(\xi)=n$, i.e.,

$$\sqrt{1-\xi^2}=\frac{1}{\mathrm{n}-2};$$

then

$$|Z_{n,1}(x)|\leqslant n \qquad \text{on} \quad [0,\xi].$$

II. The case $x\in[\xi,1]$. Denote by ξ_0 the last zero of T_{n-1}, $\xi_0=\cos(\pi/2(n-1))$. For $n\geqslant 4$

$$\sqrt{1-\xi^2}=\frac{1}{n-2}\leqslant\cos\frac{\pi}{2(n-1)}=\sqrt{1-\xi_0^2};$$

hence $\xi \geqslant \xi_0$, and therefore

$$\operatorname{sign} T'_{n-1}(x) = \operatorname{sign} T_{n-1}(x) \qquad \text{for} \quad x \in [\xi, 1].$$

This means that on $[\xi, 1]$

$$\begin{aligned} |Z_{n,1}(x)| &= \frac{1}{n-1}\, |(n-2)(x^2-1)\, T'_{n-1}(x) - T'_{n-1}(x) + (n-2)\, x T_{n-1}(x)| \\ &\leqslant \frac{1}{n-1} \max\{|(n-2)(x^2-1)\, T'_{n-1}(x) \\ &\quad - T'_{n-1}(x)|, |(n-2)\, x T_{n-1}(x)|\} \end{aligned}$$

and consequently

$$\begin{aligned} \|Z_{n,1}(x)\| &\leqslant \left\| \frac{n-2}{n-1} (1-x^2)\, T'_{n-1}(x) + \frac{1}{n-1}\, T'_{n-1}(x) \right\| \\ &\leqslant (n-2)\, \|\sqrt{1-x^2}\| + n - 1 \\ &= (n-2)\, \sqrt{1-\xi^2} + n - 1 = n; \end{aligned}$$

hence (4.11) is proved for $n \geqslant 4$. The case $n = 3$ could be verified directly. ∎

5. DUFFIN–SCHAEFFER–SCHUR TYPE INEQUALITIES

A Duffin–Schaeffer–Schur inequality (DSS-inequality) is any DS-type inequality, in which the majorant Q_n vanishes at the end points $t_0 = -1$ and $t_n = 1$. The reason is I. Shur's paper [15], where A. Markov's problem has been examined subject to zero boundary conditions. An example of DSS-inequality is given by Theorem B. For other results of a similar nature the reader may consult [5, 13, 2].

In this section we discuss the possibility for derivation of DSS-inequalities on the basis of the pointwise theorems established in Section 3. Our starting point will be property (i) of the ultraspherical polynomials. With $q = P_n^{(\lambda)}$ we have the representation

$$(x^2-1)\, q''(x) = n(n+2\lambda) \left[-\frac{2\lambda+1}{n(n+2\lambda)}\, x q'(x) + q(x) \right]. \tag{5.1}$$

Clearly, the parameter $m = -(2\lambda+1)/n(n+2\lambda)$ satisfies requirement (3.14) with equality sign; therefore Theorem 3.2 is applicable to $Q_n(x) = (x^2-1)\, q''(x)$. However, identity (5.1) will be used with respect to derivatives of Q_n, and this makes possible a formal choice of $\lambda \in [-3/2, -1/2]$. For this reason we prefer to apply Theorem 3.1 in order to prove

THEOREM 5.1. *Let* $(t_j)_1^{n-1}$ *be the zeros of* $q := P_{n-1}^{(\lambda)}$, *and let* $t_0 = -1$, $t_n = 1$. *Let* $Q_n(x) = (x^2 - 1)\, q'(x)$. *If* $f \in \pi_n$ *satisfies the inequalities*

$$|f(t_j)| \leqslant |Q_n(t_j)|, \qquad j = 0, ..., n,$$

then

$$\|f^{(k)}\| \leqslant \|Q_n^{(k)}\| \tag{5.2}$$

for each $k \in \{2, ..., n\}$, *if* $\lambda \in [0, 1/2]$, *and for* $k \in \{3, ..., n\}$, *if* $\lambda \in (-1/2, 0)$.

For these values of k *and* λ, *in* (5.2) *equality occurs only if* $f = \pm Q_n$.

Proof. To follow the notations of Theorem 3.1, we set $q = \omega_*$. Obviously, the zeros of $(x^2-1)\, \omega_*(x)$ and $Q_n(x) = (x^2-1)\, \omega'_*(x)$ interlace and repeated differentiation in (5.1) yields

$$Q_n^{(k)}(x) = [n(n+2\lambda-2) + k(1-2\lambda)]\, \omega_*^{(k-1)}(x) + (1-2\lambda)\, x\omega_*^{(k)}(x). \tag{5.3}$$

The constants $c_1 = c_1(k) = n(n+2\lambda-2) + k(1-2\lambda)$ and $c_2 = 1-2\lambda$ satisfy requirement (3.2), and we can apply Theorem 3.1 to obtain $|f^{(k)}(x)| \leqslant \max\{|Q_n^{(k)}(x)|, |Z_{n,k}(x)|\}$ for each $k \in \{1, ..., n\}$ and for every $x \in [-1, 1]$, where

$$Z_{n,k}(x) = n(n+2\lambda-2) \left[\frac{x^2-1}{k}\, \omega_*^{(k)}(x) + x\omega_*^{(k-1)}(x)\right] + (2\lambda-1)\, \omega_*^{(k)}(x). \tag{5.4}$$

For $\lambda \in [-1/2, 1/2]$ and $k \geqslant 2$, (5.3) and properties (ii)–(iii) imply

$$\|Q_n^{(k)}\| = [n(n+2\lambda-2) + k(1-2\lambda)]\, |\omega^{(k-1)}(1)| + (1-2\lambda)\, |\omega_*^{(k)}(1)|. \tag{5.5}$$

From (5.4) we get

$$\|Z_{n,k}\| \leqslant n(n+2\lambda-2) \left\|\frac{x^2-1}{k}\, \omega_*^{(k)}(x) + x\omega_*^{(k-1)}(x)\right\| + (1-2\lambda)\, \|\omega_*^{(k)}\|.$$

Then application of Lemma 2.4 implies

$$\left\|\frac{x^2-1}{k}\, \omega_*^{(k)}(x) + x\omega_*^{(k-1)}(x)\right\| = |\omega_*^{(k-1)}(1)|$$

for $k \geqslant 2$, if $\lambda \in [0, 1/2]$, and for $k \geqslant 3$, if $\lambda \in [-1/2, 0)$. Thus we have for these values of k and λ

$$\|Z_{n,k}\| \leqslant n(n+2\lambda-2)\, |\omega_*^{(k-1)}(1)| + (1-2\lambda)\, |\omega_*^{(k)}(1)|. \tag{5.6}$$

The comparison of the right-hand sides of (5.5) and (5.6) shows that $\|Z_{n,k}\| \leqslant \|Q_n^{(k)}\|$. Finally, the cases of equality in (5.2) are described by Lemma 2.3. The proof is completed. ∎

Going to the limit $\lambda \to -1/2$ in Theorems 4.1 and 5.1 one can see the validity of the following DSS-type inequalities.

THEOREM 5.2. *Let* $Q_n(x) = (1-x^2)(d/dx)\, P_{n-1}^{(1/2)}(x)$, *and let* $\{t_j\}_{j=0}^{n-2}$ *be the zeros of* $(1-x^2)(d/dx)\, P_{n-2}^{(1/2)}(x)$. *If* $f(x) = (1-x^2)\, q(x) \in \pi_n$ *and* q *satisfies*

$$|q(t_j)| \leqslant \left| \frac{d}{dx} P_{n-1}^{(1/2)}(t_j) \right| \qquad \text{for} \quad j = 0, ..., n-2,$$

then

$$\|f^{(k)}\| \leqslant \|Q_n^{(k)}\| \qquad \text{for} \quad k \in \{3, ..., n\}.$$

THEOREM 5.3. *Let* $Q_n(x) = (1-x^2)\, P_{n-2}^{(1/2)}(x)$, *and let* $\{t_j\}_{j=0}^{n-2}$ *be the zeros of* $(1-x^2)(d/dx)\, P_{n-2}^{(1/2)}(x)$. *If* $f(x) = (1-x^2)\, q(x) \in \pi_n$ *and* q *satisfies*

$$|q(t_j)| \leqslant |P_{n-2}^{(1/2)}(t_j)| \qquad \text{for} \quad j = 0, ..., n-2,$$

then

$$\|f^{(k)}\| \leqslant \|Q_n^{(k)}\| \qquad \text{for} \quad k \in \{3, ..., n\}.$$

The following is a brief explanation of how Theorems 5.2–5.3 follow from Theorems 4.1 and 5.1. For $\lambda = -1/2$, the mesh generating polynomial becomes $\omega(x) = (1-x^2)^2\, (d/dx)\, P_{n-2}^{(1/2)}(x)$. This means that at the points ± 1 the restrictions imposed on f have to be modified as follows

$$|f(\pm 1)| \leqslant |Q_n(\pm 1)| \qquad \text{and} \qquad |f'(\pm 1)| \leqslant |Q_n'(\pm 1)|. \tag{5.7}$$

For $\lambda = -1/2$ the majorants in Theorems 4.1 and 5.1 are, apart from constant factors, $Q_n(x) = (1-x^2)\, (d/dx)\, P_{n-1}^{(1/2)}(x)$ and $Q_n(x) = (1-x^2)$ $P_{n-2}^{(1/2)}(x)$, respectively. Since Q_n vanish at ± 1, f must vanish at these points, too; therefore $f(x) = (1-x^2)\, q(x)$ for some $q \in \pi_{n-2}$. Then the second inequality in (5.7) is equivalent to $|q(\pm 1)| \leqslant |(d/dx)\, P_{n-1}^{(1/2)}(\pm 1)|$ ($|q(\pm 1)| \leqslant |P_{n-2}^{(1/2)}(\pm 1)|$, respectively). Finally, the comparison of f and Q_n

at the interior zeros of $\omega(x)$ is replaced by comparison of $q(x)$ and $Q_n(x)/(1-x^2)$.

6. CONCLUDING REMARKS

1. The ingenious method of proof proposed by Duffin and Schaeffer seems hardly applicable for derivation of other DS-type inequalities. The reason is that this method exploits some special properties of the Chebyshev polynomial T_n, which are difficult to obtain for other majorants. We hope that the method described in this paper can be applied for the proof of further inequalities of DS-type.

2. Let

$$f^{(k)}(x) \approx L_n^{(k)}(f; x) := \sum_{\nu=0}^{n} l_\nu^{(k)}(x)\, f(t_\nu) \tag{6.1}$$

be the Lagrange differentiation formula based on the interpolation points $-1 := t_0 < t_1 < \cdots < t_n := 1$. If the available information $\{\tilde{f}(t_\nu)\}_{\nu=0}^{n}$ is inaccurate, and the true values $\{f(t_\nu)\}_{\nu=0}^{n}$ satisfy

$$|f(t_\nu) - \tilde{f}(t_\nu)| \leqslant \varepsilon_\nu \qquad (\nu = 0, ..., n),$$

then the exact upper bound for the roundoff error in (6.1) is given by

$$R_{n,k}^{\text{round}}[f] = \sup_{x \in [-1,1]} \left\{ \sum_{\nu=0}^{n} |l_\nu^{(k)}(x)|\, \varepsilon_\nu \right\} =: \|Q_n^{(k)}(\varepsilon; \cdot)\|,$$

where $Q_n(\varepsilon; \cdot) = Q_n(\varepsilon_0, ..., \varepsilon_n; \cdot)$ is the extremal polynomial in the DS-type inequality related to $\Delta = \{t_j\}_{j=0}^{n}$ and $\varepsilon = (\varepsilon_0, \varepsilon_1, ..., \varepsilon_n)$. This indicates that the inequalities of DS-type are also of some practical interest.

3. Concerning DS-inequalities, some questions arise in a natural way. Such a question is, for a fixed majorant Q_n, what is the set of all meshes Δ admitting DS-type inequality? As we already mentioned, the original DS-inequality fails to hold if in (1.2) some of the points η_j^n are omitted. However, is it not true that $\Delta = \{\eta_j^n\}_{j=0}^{n}$ is the unique mesh allowing DS-type inequality with $Q_n = T_n$. A trivial alternative choice is any $n+1$-tuple, containing the zeros of T_n. Theorem 4.2 provides another, non-trivial mesh. More generally, Theorem D and Theorem 4.1 assert that, for $Q_n = P_n^{(\lambda)}$ DS-inequality holds for two choices of a mesh $\Delta = \{t_j\}$, namely for $\{t_j\}_1^{n-1}$ being the zeros of $P_{n-1}^{(\lambda+1)}$ and $P_{n-1}^{(\lambda)}$. We conjecture that if $\lambda \geqslant 0$, then DS-inequality holds with $Q_n = P_n^{(\lambda)}$ for any choice of $\{t_j\}_1^{n-1}$—the zeros of $P_{n-1}^{(\mu)}$ with $\lambda \leqslant \mu \leqslant \lambda + 1$.

The converse question is, for a given mesh Δ (i.e., a set of $n+1$ distinct points, located in $[-1, 1]$), what is the class of all majorants Q_n at these points, admitting Duffin and Schaeffer type inequality? Theorems 3.1 and 3.2 give some possible candidates for such majorants. In particular, Theorem B and Theorem 4.2 show that the polynomials $(1-x^2)\, U_{n-2}(x)$ and $T_n(x)$ are extremal with respect to the mesh Δ formed by ± 1 and the zeros of T_{n-1}.

4. The special case $\lambda = 1/2$ in Theorem 5.1 corresponds to Theorem D ($\lambda = -1/2$) (see also [2, Theorem 3.2]), while Theorem 5.1 ($\lambda = 0$) reproduces Theorem B. Theorem 5.3 is close in spirit to the result in [13, Theorem 1], where $Q_n(x) = (1-x^2)\, T_{n-2}(x)$ and $\omega(x) = (1-x^2)^2\, T'_{n-2}(x)$.

5. Lemma 2.4 is the easiest but not the only way for obtaining DS-type inequalities from those pointwise. To prove extremality of Q_n, based on the pointwise theorems in Section 3, it suffices to show that

$$\|Z_{n,k}\|_{C(J_{n,k})} \leqslant \|Q_n^{(k)}\|,$$

and this could be valid even if

$$\|Z_{n,k}\| > Z_{n,k}(1), \qquad \text{or} \qquad Z_{n,k}(1) > Q_n^{(k)}(1),$$

i.e., when Lemma 2.4 is not applicable. In the latter case the observation

$$|Z_{n,k}(x)| \leqslant |Q_n^{(k)}(x)| \qquad \text{for} \quad x \in [\beta_{n-k}^k, x_0]$$

may turn out to be useful. Namely, one can try to prove that $\|Z_{n,k}\|_{C[-x_0, x_0]} = |Z_{n,k}(x_0)|$ (note that $x_0 < 1$ in this case).

6. Theorem 3.3 may be applied for derivation of some DS-inequalities with non-symmetric majorants, e.g., for $Q_n = P_n^{(\alpha,\beta)}$—the Jacobi orthogonal polynomials. One can also formulate and prove without any difficulties a nonsymmetric version of Theorem 3.1.

7. Finally note that the inequalities of DS and DSS type in Sections 4 and 5 remain valid for the class of polynomials with complex coefficients. This follows easily from the fact that the class of polynomials into consideration is invariant with respect to the operations: (i) multiplication by $e^{i\theta}$, θ is real; (ii) taking the real part.

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